



Every subset of $\mathbb{R}$ is bounded above. Here is a proof. The environment Δ has three primitive definitions and one descriptive one; a subset of $\mathbb{R}$ is a predicate $V : \mathbb{R} \rightarrow *p$.

$$\begin{aligned} \mathbb{R} &:= \perp\!\!\perp : *s & \leq &:= \perp\!\!\perp : \mathbb{R} \rightarrow \mathbb{R} \rightarrow *p & le_refl &:= \perp\!\!\perp : \forall x:\mathbb{R}. x \leq x \\ V:\mathbb{R} \rightarrow *p &\triangleright bounded(V) &:= \forall x:\mathbb{R}. \exists y:\mathbb{R}. (Vx \Rightarrow x \leq y) : *p \end{aligned}$$

The derivation is correct: every line follows by a rule, and the kernel accepts it. Parameter lists are suppressed.

(a)	$V : \mathbb{R} \rightarrow *p$	
(b)	$x : \mathbb{R}$	
(c)	$h : Vx$	
(1)	$a_1 := le_refl\ x : x \leq x$	$\forall\text{-el on } le_refl$
(2)	$a_2 := \lambda h:Vx. a_1 : Vx \Rightarrow x \leq x$	$\Rightarrow\text{-in on (1)}$
(3)	$a_3 := \exists\text{-in}(\mathbb{R}, \lambda y. (Vx \Rightarrow x \leq y), x, a_2) : \exists y:\mathbb{R}. (Vx \Rightarrow x \leq y)$	$\exists\text{-in}$
(4)	$a_4 := \lambda x:\mathbb{R}. a_3 : \forall x:\mathbb{R}. \exists y:\mathbb{R}. (Vx \Rightarrow x \leq y)$	$\forall\text{-in on (3)}$
(5)	$p := a_4 : bounded(V)$	$(conv)$

- (a) All of $\mathbb{R}$ has no upper bound, yet line (5) proves it bounded. Why does the kernel accept this? Where exactly is the mistake?
- (b) Correct it. Then go through (1)–(5) again: which lines still type-check, and at which line, and why, does the proof break?

The same in Lean; the audit lists `le_refl` only.

```
axiom R : Type
axiom le : R → R → Prop
axiom le_refl : ∀ x : R, le x x
def Bounded (V : R → Prop) : Prop := ∀ x : R, ∃ y : R, V x → le x y
theorem every_set_bounded (V : R → Prop) : Bounded V :=
  fun x => {x, fun _ => le_refl x}
#print axioms every_set_bounded
```

lines (1)–(4)
[le_refl]

Lean: Exercises.lean, section 10.

I Definitions

- **Unfolding.** $\Gamma \triangleright a(\bar{x}) := M : N$ unfolds by δ : $a(\bar{U}) \rightarrow_{\delta} M[\bar{x} := \bar{U}]$. $(conv)$: a term of type B may be used at any $B' =_{\beta\delta} B$.
- $\forall\text{-el}$: from $u : \forall x:S. Px$ and $a : S, u a : Pa$.
- $\forall\text{-in}$: from $M : Px$ under a flag $x : S, \lambda x:S. M : \forall x:S. Px$, closing the flag. $\Rightarrow\text{-in}$: the same, with P not depending on x .
- $\exists\text{-in}(S, P, a, u) : \exists x:S. Px$ from $a : S$ and $u : Pa$.

Name _____

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Answer

Out of room? Continue on the back of the page.